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Optimal Direction of Grasped Object Minimizing Contact Forces

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***Abstract*— When grasping an object, friction forces are sometimes utilized. Because of this friction forces, the object is manipulable to various direction. In this paper, we discuss which direction is best when contact points, the number of which is two in the 2D-space grasping or three in the 3D one, are assigned. To evaluate the object direction, we focus on the norm of contact forces consisting of the normal forces and friction forces. We conclude that the norm of contact force become minimal when vertical line through the center of mass of the object (i.e., gravitational line) shoots out the midpoint of two contact points in 2D space, while it does the centroid of the contact-point triangles in 3D space.**

Index Terms* – *Grasping, Contact forces, Object direction, Optimization

I. Introduction

Grasping is one of important motions for robots to achieve given tasks such as conveying, assembling, and manipulating objects. When grasping an object, humans sometimes utilize friction forces. Although this grasping method lacks the reliability from the aspect of fixing the object in comparison with the power grasping method [1], it has a possibility to manipulate the object speedy and skillfully.

From this point of view, we consider the following problem for grasping using friction.

- When the contact points are assigned on the object surface for grasping, which direction of the object is optimal.

There are many optimizing factors for grasping because of the redundancy of this task: position of contact points, internal forces, finger joint torques and so on [2]. Especially, the problem of contact points is significant for automatic realization of grasping tasks, and so many studies treat this problem[3], [4], [5]. However, we often encounter cases where the position of the grasping points are restricted. For example, when we paint the object, limited parts of the surface are allowed for touching. If a part of object is fragile, other rigid parts must be chosen for grasping. Mechanical conditions such as the size of the hand are also the factors that restrict the contact points for large grasped objects. Putting these cases together, we set the problem for grasping so that a prehensible set of contact points are assigned.

Some optimizing factors such as finger joint torques depend on the structure of the end-effectors. However, there are some factors which are independent of the end-effectors. One of them is the direction of the grasped object. Actually, there will be a case where the attitude of the object must be maintained, e.g., moving a cup which is filled with liquid. In many cases, however, we can select their attitude or direction freely. Thus, the direction of the grasping object is a problem to be solved for realizing a automatic handling, but the optimization of the grasped object is not discussed so much in the previous studies.

For the first step, we here treat a static grasping with friction. As a evaluation of the grasping, the norm of the contact forces is selected, because smaller contact forces not only achieves an efficient grasping by use of the small grasping forces but also provides less possibility for object to be broken. Throughout this paper, we set the following assumptions on the object.

- An object is rigid.
- The positional relation between the center of mass and contact points is known.
- The contact is limited to the point contact with friction[6].
- At the contact points, the shape of the object is smooth.
- Sufficient large frictions act on the surface of the object.

Note that the last assumption implies that the friction corn conditions are not considered. Namely, the problem here is what will be the optimal direction of grasped object if this object is prehensible by the given contact points.

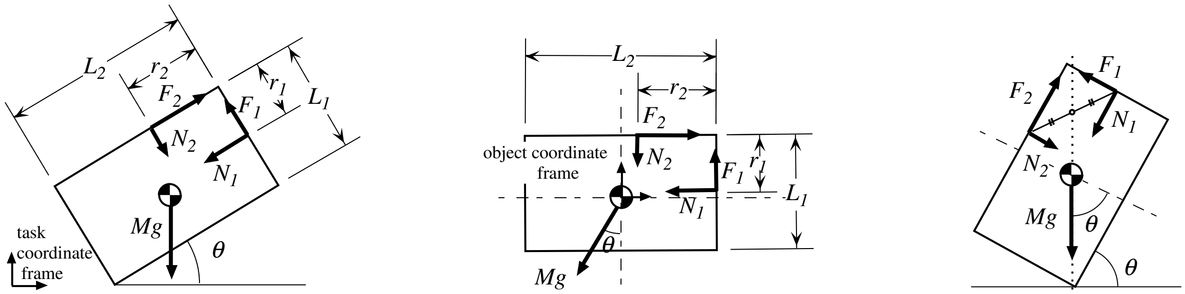
Firstly, we consider a case of the grasping in the 2D plane including the gravitational direction. Next we extend the result here to three-dimensional case. As for the number of the contact points, the least case is considered here: two contact points in the 2D grasping and three in the 3D grasping. Indeed, more contact points make it easier to grasp an object. However, the less contact points will be required when manipulating it within one end-effector (hand).

II. 2D Grasping in the vertical plane

To begin with, we introduce a simple case study where the object is a rectangle. Next, we extend the conclusion here to any convex objects.

A. Rectangular object

The homogeneous rectangular is grasped by two points on the adjacent edges using the friction. As shown in Fig. 1(a), the distance from one vertex to contact points are $r_1(< L_1)$ and $r_2(< L_2)$, where L_1 and L_2 is the length of the edges of the rectangular object. The problem is to find attitude angle



(a) Object in the task space. (b) The object coordinate frame. (c) The Optimal grasping

Fig. 1. Grasping of a rectangular object

of object from the gravitational direction under the evaluation on the contact forces.

This problem becomes accessible by means of introducing the object coordinate frame as shown in Fig. 1(b). The origin of the object frame is set to the center of mass, and two coordinate axes are parallel to two edges containing the contact points. The object attitude in the task coordinate frame is expressed as the direction of the gravity in the object frame, which is denoted as the clockwise angle θ from the y axis. To restricting the grasping only to the pinching up one, we can restrict θ in the range $0 \leq \theta \leq \pi/2$.

The balance of the forces and moments in the stable grasping are described by the equation,

$$L\boldsymbol{F} = \boldsymbol{M} \tag{1}$$

Here, L is a grasping matrix given as

$$L = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ \frac{L_1}{2} - r_1 & \frac{L_2}{2} & -\frac{L_2}{2} + r_2 & -\frac{L_1}{2} \end{bmatrix} \tag{2}$$

$\boldsymbol{F}$ is a unknown contact force vector whose components are the normal force N_i and friction force F_i (i=1,2), i.e.,

$$\boldsymbol{F} = \begin{bmatrix} N_1 & F_1 & N_2 & F_2 \end{bmatrix}^T \tag{3}$$

and $\boldsymbol{M}$ is given as

$$\boldsymbol{M} = \begin{bmatrix} -Mg\sin\theta & -Mg\cos\theta & 0 \end{bmatrix}^T \tag{4}$$

representing the action of the gravity, where M is a mass of the object and g is a constant of gravitation.

The solution of the equation (1) can be written as

$$\boldsymbol{F} = \boldsymbol{F}_T(\theta) + \alpha\boldsymbol{F}_N \tag{5}$$

where

$$\boldsymbol{F}_T(\theta) = L^\dagger \boldsymbol{M}(\theta) \tag{6}$$

$L^\dagger$ is a pseudo-inverse matrix of L [7], $\boldsymbol{F}_N$ is a unit vector in the null space of L, i.e.,

$$L\boldsymbol{F}_N = \boldsymbol{0} \tag{7}$$

and α is a scalar corresponding to the amount of the internal forces.

Among the solutions of the equation (1), we select the one which minimizes the following evaluation function V

$$V = \boldsymbol{F}^T\boldsymbol{F}. \tag{8}$$

Substituting (8) for (5), we obtain

$$V = ||\boldsymbol{F}_T(\theta)||^2 + \alpha^2||\boldsymbol{F}_N||^2 \tag{9}$$

Apparently, the smaller the α is, the better this evaluation function becomes.

Regarding to the object direction, we differentiate V with respect to θ

$$\begin{aligned} \frac{\partial V}{\partial \theta} &= \frac{\partial}{\partial \theta}\left\{\boldsymbol{M}^T(\theta)(L^\dagger)^T L^\dagger \boldsymbol{M}(\theta)\right\} \\ &= \big((-2L_1L_2 + 2L_1r_2 + 2r_1L_2 - 2r_1r_2)\cos^2\theta \\ &\quad +((L_1 - r_1)^2 - (L_2 - r_2)^2)\cos\theta\sin\theta \\ &\quad +L_1L_2 - L_1r_2 - r_1L_2 + r_1r_2\big)\frac{Mg^2}{(r_1^2 + r_2^2)} \end{aligned} \tag{10}$$

Solving the equation $\frac{\partial V}{\partial \theta} = 0$ in the range $0 \leq \theta \leq \pi/2$, the minimum point of V is given as

$$\theta^* = \tan^{-1}\left(\frac{L_2 - r_2}{L_1 - r_1}\right). \tag{11}$$

The object attitude of this solution is illustrated in Fig. 1(c).

B. Convex object with smooth shape

The shape of object is assumed to be given as a known function $f(x, y) = 0$ in the object coordinate frame. The function $f(x, y)$ is assumed differentiable at the given two contact points $\boldsymbol{p}_1 = (x_1, y_1)^T$ and $\boldsymbol{p}_2 = (x_2, y_2)^T (\neq \boldsymbol{p}_1)$. The question is which direction of the object makes contact forces minimal. Note that, $f(x_1, y_1) = f(x_2, y_2) = 0$ are satisfied.

The result of the previous case study implies that, at the optimal attitude, the midpoint of two contact points lies on the vertical line through the center of mass of the object. Actually, it can be generalized as the following theorem:

Theorem 1: Consider the case when the convex object is grasped in the planer plane including the gravitational direction with two contact points. Then, the square norm of contact forces takes minimum when the midpoint of two contact points and the center of mass of the object are aligned on the gravitational direction.

Proof: As is the same as in the previous section, the effect of the gravity is expressed in the object coordinate frame. Setting the direction of the gravity θ clockwise from the negative direction of y axis in the object coordinate frame, this effect is denoted by a vector $\boldsymbol{M}$ as

$$\boldsymbol{M} = \boldsymbol{M}(\theta) = \begin{bmatrix} -Mg\sin\theta & -Mg\cos\theta & 0 \end{bmatrix}^T \quad (12)$$

Here, we try to find the optimal θ that minimizes the norm of contact forces.

In order to consider the effect of the object shape, we put the angle between the normal line at the contact point and x axis of the object frame to ϕ_i as shown in Fig. 2(a). This ϕ_i is uniquely determined from the object shape $f(x, y) = 0$ as

$$\phi_i = \arctan 2\left(\left(\frac{\partial f}{\partial y}\right)_{p_i}, \left(\frac{\partial f}{\partial x}\right)_{p_i}\right). \quad (13)$$

Then, the balance of the contact forces (3) and gravitational force (12) is expressed as the same equation (1) as the case of the rectangular object, where the grasping matrix L is described with ϕ as

$$L = \begin{bmatrix} \cos\phi_1 & -\sin\phi_1 & \cos\phi_2 & -\sin\phi_2 \\ \sin\phi_1 & \cos\phi_1 & \sin\phi_2 & \cos\phi_2 \\ L_{31} & L_{32} & L_{33} & L_{34} \end{bmatrix} \quad (14)$$

$$L_{31} = -x_1\sin\phi_1 + y_1\cos\phi_1 \quad (15)$$
$$L_{32} = -x_1\cos\phi_1 - y_1\sin\phi_1 \quad (16)$$
$$L_{33} = -x_2\sin\phi_2 + y_2\cos\phi_2 \quad (17)$$
$$L_{34} = -x_2\cos\phi_2 - y_2\sin\phi_2 \quad (18)$$

We solve the equation (1) under the evaluation function (8).

This solution can be described in the form as (5). Since $\frac{\partial V}{\partial \alpha} = 0$ implies $\alpha = 0$, the smaller internal force provides the better evaluation. However, our concern is the object direction θ that minimizes V. Using (6), (7) and the relation

$$L^{\dagger} = L^T(LL^T)^{-1} \quad (19)$$

the evaluation function becomes

$$V = \boldsymbol{M}^T(LL^T)^{-1}\boldsymbol{M} + \alpha^2 \quad (20)$$

where

$$LL^T = \begin{bmatrix} 2 & 0 & -Y \\ 0 & 2 & X \\ -Y & X & \lambda \end{bmatrix} \quad (21)$$

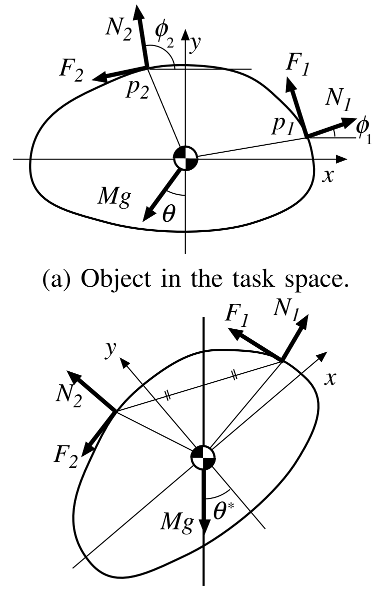
$$X = x_1 + x_2 \quad (22)$$

$$Y = y_1 + y_2 \quad (23)$$

$$\lambda = x_1^2 + x_2^2 + y_1^2 + y_2^2 \quad (24)$$

Calculating $(LL^T)^{-1}$ and substituting (20), we finally obtain

$$V = \frac{(Mg)^2}{\Delta}(2\lambda - [X\sin\theta + Y\cos\theta]^2) + \alpha^2 \quad (25)$$



(b) optimal grasping

Fig. 2. Grasping of a convex object

where

$$\Delta = 2((x_1 - x_2)^2 + (y_1 - y_2)^2) \neq 0 \quad (26)$$

At the optimal posture $\theta = \theta^*$,

$$\begin{bmatrix} X \\ Y \end{bmatrix} /\!/ \begin{bmatrix} \sin\theta^* \\ \cos\theta^* \end{bmatrix} \quad (27)$$

is satisfied. Because the midpoint of two contact points is $(X/2, Y/2)$, the vector $[\sin\theta^*, \cos\theta^*]^T$ points to the opposite direction of the gravity, as illustrated in Fig. 2(b). Thus, the theorem have been proved. ∎

Note 1: This result is satisfied regardless of the object shape. And, when the midpoint of two contact points coincides the center of mass of the object, both X and Y becomes zero. Then, the norm of the contact forces become the same regardless of the object direction.

III. Grasping in 3D space

The theorem in the previous section is applicable only to the grasping in 2D space. Here, we extend it 3D space.

A. Assumptions and problem setting

We set the following assumptions on the shape of the grasped object:

- The shape of an object is assumed to be expressed by the known convex function $f(x, y, z) = 0$ in the object coordinate frame O-XYZ whose origin is set to the center of mass of this object.
- Three contact points are given as $\boldsymbol{p}^i = (x^i, y^i)^T (i = 1, 2, 3)$, which are not aligned on the same straight line.
- The shape of the contact points are smooth, i.e., $f(x, y, z)$ is differentiable at $\boldsymbol{p}^i$.

Then, the question is which attitude of the grasped object makes the contact forces minimal. This attitude in the task coordinate frame is represented as the direction of the gravity in the object coordinate frame. We express this direction using two parameters ϕ and θ, where ϕ $(0 \le \phi \le \pi/2)$ is the angle between the gravitational direction and the negative direction of Z axis, and θ is the azimuthal angle of the gravity from the X axis in the X-Y plane, as illustrated in Fig. 3. Putting the contact force vector to $\boldsymbol{f}$, the problem is formulated as follows: Find the optimal ϕ and θ that minimize the evaluation function

$$V = \boldsymbol{f}^T \boldsymbol{f} \tag{28}$$

from the set of the angles satisfying the static balance of grasping.

B. Analysis

In order to express the contact forces, we introduce the contact-point coordinate frame O^i-$X^iY^iZ^i$ whose origin is set the contact point $\boldsymbol{p}^i$. As shown in Fig. 3 the Z^i is set to the normal direction of the surface at $\boldsymbol{p}^i$, i.e., $(\partial f/\partial x, \partial f/\partial y, \partial f/\partial z)_{\boldsymbol{p}^i}$ and the Y^i axis is set to the direction orthogonal to Z^i axis within the plane containing both the Z^i axis and the line through $\boldsymbol{p}^i$ parallel to the Z axis. The X^i axis is set so that O^i-$X^iY^iZ^i$ become the right-handed coordinate system. The relation between the object and contact-point coordinate frame is expressed by two parameters ξ^i and η^i, where ξ^i is the angle between the Z axis and the Z^i axis, and η^i is the azimuthal angle of the Z^i axis from the X axis in the X-Y plane, as shown in Fig. 3

The contact force is expressed in the coordinate frame O^i-$X^iY^iZ^i$ as $\boldsymbol{f}^i = (f^i_x, f^i_y, f^i_z)^T$, where the magnitude of normal force and the friction are respectively f_z and $\sqrt{(f^i_x)^2 + (f^i_y)^2}$. $\boldsymbol{f}^i$ can be transformed to the contact force vector in the object coordinate frame, i.e., $\boldsymbol{F}^i = (F^i_x, F^i_y, F^i_z)^T$ using the transform matrix T^i:

$$\boldsymbol{F}^i = T^i \boldsymbol{f}^i \tag{29}$$

where

$$T^i = \begin{bmatrix} -\sin\eta^i & -\cos\xi^i\cos\eta^i & \sin\xi^i\cos\eta^i \\ \cos\eta^i & -\cos\xi^i\sin\eta^i & \sin\xi^i\sin\eta^i \\ 0 & \sin\xi^i & \cos\xi^i \end{bmatrix} \tag{30}$$

Now, we consider the static balance of the grasping object in the object coordinate frame. The contact force vector $\boldsymbol{F}$ in the object frame is defined as $\boldsymbol{F} = [\boldsymbol{F}^{1^T}, \boldsymbol{F}^{2^T}, \boldsymbol{F}^{3^T}]^T$. Then, the static balance is described as

$$L\boldsymbol{F} = \boldsymbol{M} \tag{31}$$

Here, $\boldsymbol{M}$ denote the action of gravity

$$\boldsymbol{M} = \begin{bmatrix} M_x & M_y & M_z & 0 & 0 & 0 \end{bmatrix}^T \tag{32}$$

$$M_x = Mg\cos\theta\sin\phi \tag{33}$$

$$M_y = Mg\sin\theta\sin\phi \tag{34}$$

$$M_z = Mg\cos\theta \tag{35}$$

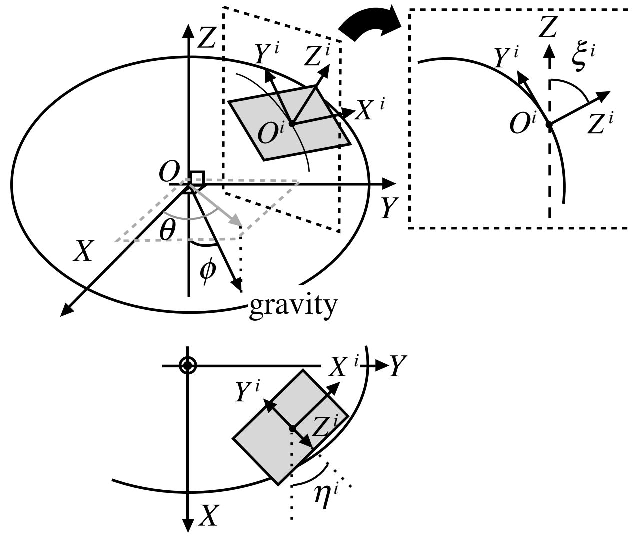


Fig. 3. Contact-point coordinate frame.

The Matrix $L \in \boldsymbol{R}^{6\times9}$

$$L = \begin{bmatrix} I_3 & I_3 & I_3 \\ R_1 & R_2 & R_3 \end{bmatrix} \tag{36}$$

where $I_3 \in \boldsymbol{R}^{3\times3}$ is a unit matrix and $R_i \in \boldsymbol{R}^{3\times3}$ is a skew-symmetrical matrix determined by the position of the contact point:

$$R_i = \begin{bmatrix} 0 & -z_i & y_i \\ z_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix}. \tag{37}$$

The solution of (31) is represented with the pseudo-inverse matrix of L, i.e., $L^\dagger$ [7] as

$$\boldsymbol{F} = L^\dagger \boldsymbol{M} + (L^\dagger L - I_9)\boldsymbol{p} \tag{38}$$

where $I_9 \in \boldsymbol{R}^{9\times9}$ is a unit matrix, and $\boldsymbol{p} \in \boldsymbol{R}^9$ is an arbitrary vector. The second term can be expressed using the orthogonal unit vectors $\boldsymbol{e}_i \in \boldsymbol{R}^6 (i = 1, 2, 3)$ that spanning the null space of L as

$$(L^\dagger L - I_9)\boldsymbol{p} = L_N \boldsymbol{\alpha} \tag{39}$$

Here, $L_N = \begin{bmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \end{bmatrix}$ and $\boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}^T$. Each α_i correspond to the magnitude of the internal force. Accordingly, the solutions of the equation (31) are represented in the form as

$$\boldsymbol{F} = (L^\dagger \boldsymbol{M} + L_N \boldsymbol{\alpha}) \tag{40}$$

Among these solutions, we find the optimal one that minimizes the evaluation function (28). Here, the contact vector in the coordinate frame O^i-$X^iY^iZ^i$, i.e., $\boldsymbol{f} = \begin{bmatrix} \boldsymbol{f}^1 & \boldsymbol{f}^2 & \boldsymbol{f}^3 \end{bmatrix}^T$ becomes

$$\boldsymbol{f} = T^{-1}(L^\dagger \boldsymbol{M} + L_N \boldsymbol{\alpha}) \tag{41}$$

where

$$T = diag[T^1, T^2, T^3] \tag{42}$$

Thanks to the orthogonality between $L^{\dagger}$ and L_N as well as $T^{-1} = T^T$, (28) becomes

$$V = \boldsymbol{M}^T (LL^T)^{-1} \boldsymbol{M} + ||\boldsymbol{\alpha}||^2 \tag{43}$$

From the same discussion as the previous section, the smaller $\boldsymbol{\alpha}$ provides the better evaluation. So, the subsequent analysis is limited to find the optimal angle ϕ and θ. However, the straightforward analysis is difficult due to the high dimension. Thus, we firstly examine some case studies to make an induction of a general conclusion.

1) Contact points constructing an equilateral triangle: Consider the case where the contact points are given as follows:

$$\boldsymbol{p}_1 = (\ 6 \quad -2\sqrt{3} \quad 4\ )^T \tag{44}$$

$$\boldsymbol{p}_2 = (\ -6 \quad -2\sqrt{3} \quad 4\ )^T \tag{45}$$

$$\boldsymbol{p}_3 = (\ 0 \quad 4\sqrt{3} \quad 4\ )^T \tag{46}$$

Then, the evaluation function becomes

$$V = \frac{(Mg)^2}{9}(5 - \cos\phi) \tag{47}$$

Thus, the angles that minimize this evaluation function, i.e., the solution of $\frac{\partial V}{\partial \phi} = 0$ is:

$$\phi = 0 \tag{48}$$

This result implies that the line of gravity from the center of mass of the object runs through any one of the incenter, the circumcenter, the orthocenter or the centroid of the triangle composed by three contact points.

2) Contact points constructing an isosceles right triangle: Consider the case where the contact points are given as follows:

$$\boldsymbol{p}_1 = (\ 4\sqrt{3} \quad 0 \quad 4\ )^T \tag{49}$$

$$\boldsymbol{p}_2 = (\ -4\sqrt{3} \quad 0 \quad 4\ )^T \tag{50}$$

$$\boldsymbol{p}_3 = (\ 0 \quad 4\sqrt{3} \quad 4\ )^T \tag{51}$$

Then, the evaluation function becomes

$$V = \frac{(Mg)^2}{24}(13\cos^2\theta\sin^2\phi + 20\sin^2\theta\sin^2\phi - 8\sqrt{3}\sin\theta\sin\phi\cos\phi + 12\cos^2\phi) \tag{52}$$

For minimization, put the derivative of V to zero as

$$\frac{\partial V}{\partial \theta} = \frac{(Mg)^2}{24}(14\cos\theta\sin\theta\sin^2\phi - 8\sqrt{3}\cos\theta\sin\phi) = 0 \tag{53}$$

$$\frac{\partial V}{\partial \phi} = -\frac{(Mg)^2}{24}(7\cos^2\theta\sin\phi\cos\phi - 8\cos\phi\sin\phi + 8\sqrt{3}\sin\theta\cos^2\phi - 4\sqrt{3}\sin\theta) = 0 \tag{54}$$

Solving the above two simultaneous equations, we obtain the solution

$$(\theta, \phi) = \left(-\frac{\pi}{2}, \frac{\pi}{6}\right) \tag{55}$$

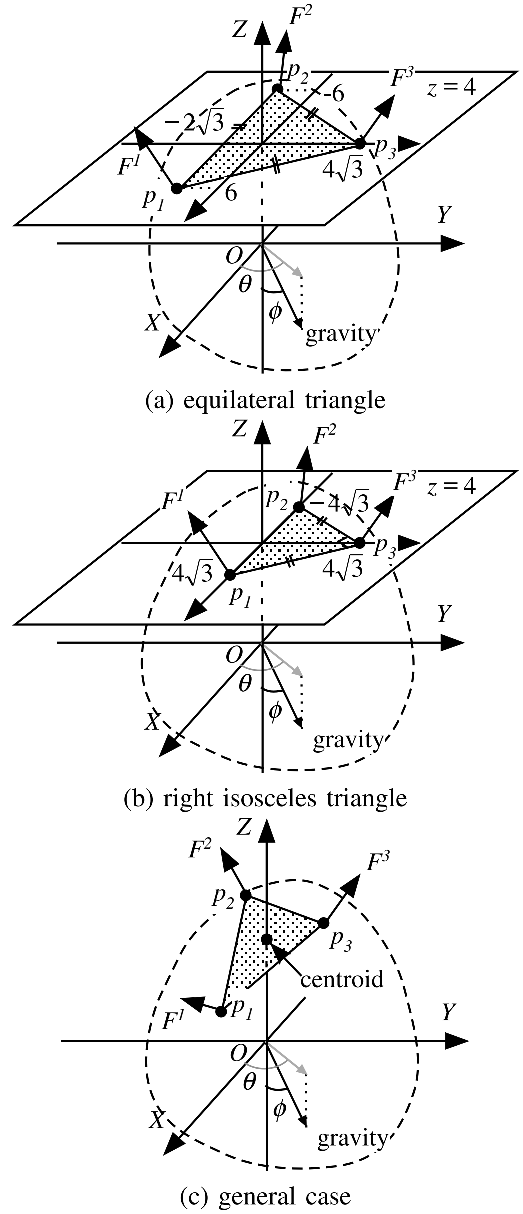


Fig. 4. Examples of contact-point triangle.

This result implies that the line of gravity from the center of mass of the object runs through the centroid of the triangle composed by three contact points.

3) General case: The above two case studies provide a suggestive result: if the gravity passes the centroid of the triangle composed by three contact points, then contact forces becomes minimal. However, the above two cases are specific in that the contact points construct the equilateral or isosceles right triangle. In addition, the line passing the centroid of the contact-point triangle is orthogonal to the plane containing this triangle.

Thus, we try to remove these special conditions. At first, we reset the object coordinate frame so that the Z axis passes the centroid of the contact-point triangle. This new coordinate system are used in the remaining section,

In the new coordinate frame, the contact points are given as follows:

$$\boldsymbol{p}_1 = (\ x_1 \quad y_1 \quad z_1\ )^T \tag{56}$$
$$\boldsymbol{p}_2 = (\ x_2 \quad y_2 \quad z_2\ )^T \tag{57}$$
$$\boldsymbol{p}_3 = (\ x_3 \quad y_3 \quad z_3\ )^T \tag{58}$$

Here, owing to the definition of the coordinate system, the following relations are satisfied;

$$x_1 + x_2 + x_3 = 0 \tag{59}$$
$$y_1 + y_2 + y_3 = 0 \tag{60}$$
$$z_1 + z_2 + z_3 > 0 \tag{61}$$

Here, (61) is a condition for pinching up grasping, because, in this case, the centroid of contact-point triangles must be located at the higher position that the center of mass.

Then, the matrix LL^T in (43) becomes

$$LL^T = \begin{bmatrix} 3 & 0 & 0 & 0 & Z & 0 \\ 0 & 3 & 0 & -Z & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & -Z & 0 & \alpha & \lambda & \kappa \\ Z & 0 & 0 & \lambda & \beta & \nu \\ 0 & 0 & 0 & \kappa & \nu & \gamma \end{bmatrix} \tag{62}$$

$$Z = z_1 + z_2 + z_3 \tag{63}$$
$$\alpha = y_1^2 + y_2^2 + y_3^2 + z_1^2 + z_2^2 + z_3^2 \tag{64}$$
$$\beta = z_1^2 + z_2^2 + z_3^2 + x_1^2 + x_2^2 + x_3^2 \tag{65}$$
$$\gamma = x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2 \tag{66}$$
$$\lambda = -x_1y_1 - x_2y_2 - x_3y_3 \tag{67}$$
$$\kappa = -z_1x_1 - z_2x_2 - z_3x_3 \tag{68}$$
$$\nu = -y_1z_1 - y_2z_2 - z_3y_3 \tag{69}$$

Then, the evaluation function becomes

$$\begin{aligned} V \quad = \quad & \frac{(Mg)^2}{E_0}(E_1 \cos^2\theta \sin^2\phi \\ & + E_2 \cos^2\phi + E_3 \cos\theta \sin\theta \sin^2\phi + E_4) \end{aligned} \tag{70}$$

Here, E_k $(k = 0, \cdots, 4)$ is a constant which determined by x_i, y_i, z_i $(i = 1, 2, 3)$. Calculating $\frac{\partial V}{\partial \phi} = 0$ and $\frac{\partial V}{\partial \theta} = 0$, we obtain the following equations

$$\frac{(Mg)^2 \sin^2\phi}{E_0}(-E_1 \sin 2\theta + E_3 \cos 2\theta) = 0 \tag{71}$$

$$\frac{(Mg)^2 \sin\phi \cos\phi}{E_0}(E_1 \cos 2\theta + E_3 \sin 2\theta + E_1 - 2E_2) = 0 \tag{72}$$

The two cases are possible for the solution, i.e., $\sin\phi = 0$ or

$$-E_1 \sin 2\theta + E_3 \cos 2\theta = 0 \tag{73}$$

$$E_1 \cos 2\theta + E_3 \sin 2\theta + E_1 - 2E_2 = 0. \tag{74}$$

However, the latter rarely hold since $\sin 2\theta$ and $\cos 2\theta$ must satisfy the third equation:

$$\sin^2 2\theta + \cos^2 2\theta = 1 \tag{75}$$

Thus, generally speaking, the solution of the above two simultaneous equations is

$$\phi = 0 \qquad (\theta \text{ is arbitrary}) \tag{76}$$

When $\phi = 0$, the direction of gravity coincide with the Z axis, which pass the centroid of the contact-point triangle.

From this result, we obtain the following theorem.

Theorem 2: The prehensible set of three contact points are assigned on the surface of the object in the 3D space. The object is assumed to be convex, and to be smooth at these contact points. Then, the norm of the contact forces takes minimum at the object attitude where the centroid of contact-point triangle and the center of mass of the object are aligned on the gravitational direction.

IV. Conclusion

In this paper, we treated the following problem on the object grasping: when a prehensible set of contact points are assigned for grasping, which attitude angles make contact forces small. Assuming the convexity of the object, its smooth shape at the contact points and large friction on its surface, we derived the following results on the optimal object attitude that minimizes the square norm of contact forces:

- In 2D space, the midpoint of two contact points and the center of mass of the object are aligned on the gravitational direction.
- In 3D space, the centroid of contact-point triangle and the center of mass of the object are aligned on the gravitational direction.

In the analysis, we did not treat friction corns well. As a future work, we extend our result to consider them as well as the roll of the inertial forces.

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