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Mathematical Analysis on the Relation between Object Orientation and Contact Forces in the 2D Grasp with Frictions

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Abstract: This paper reports an analysis of 2D grasp where a convex rigid object is grasped by two contact points with frictions. The purpose is to find the object orientation that minimizes the norm of the contact force vector, each element of which is composed from the normal and friction force at each contact point. The formulation of this problem includes some equality or inequality conditions. In the analysis, the solution of the equality conditions is parameterized at first. Based on the fact that the norm of the contact force vector becomes monotonic increasing function of this parameter, the minimal parameter values are calculated by means of the piecewise analysis. Using the relation between the friction coefficient and the apex angle of the friction cone effectively, the following result is obtained: the norm of the contact force takes a local minimal value at the situation that the intersection point of the upper sides of each friction cones at two contact points is located in the opposite direction of the gravity from the center of mass of the grasped object.

Keywords: Grasping, object posture, contact force, minimization

1. INTRODUCTION

Frictions are important to grasp an object. Although the grasp with friction would not be more reliable than the one based on the Form-Closure (Bicchi (1995)), it contains the possibility for skillful manipulation of the grasped object. From this point of view, the grasp with frictions are focused on in this paper.

Many factors must be determined in the planning of the grasp: position of the contact points, magnitude of the internal force, joint torques of the grasping mechanisms, and so on. Among them, the orientation of the grasped object is addressed. In some tasks like a material handling operation, the object posture can be chosen arbitrarily. Thus, the determination of the optimal object orientation is a meaningful issue from the viewpoint of the task planning. Many papers treat the grasping problem from the aspect of the optimization (Shimoga (1996); Markenscoff and Papadimitriou (1989); Mirtich and Canny (1994); Mangialardi et al. (1996); Watanabe and Yoshikawa (2003); Cheng and Orin (1990); Liu (1999); Ding et al. (2001); Al-Gallaf (2006); Dubey et al. (1999)). However, the object orientation is seldom selected as the optimization factors.

In this paper, the relation between object orientation and the contact forces are discussed to find the object orientation that requires less contact forces.

2. GRASP WITH FRICTIONS

2.1 Assumptions

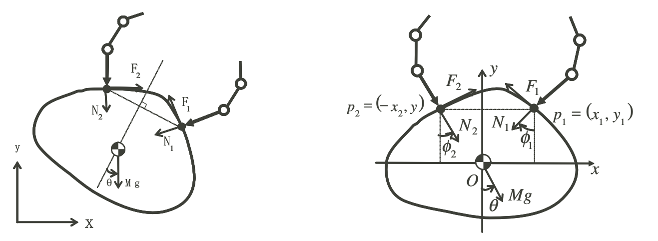
The following assumptions allow us to solve the problem here in an analytical, not numerical, manner.

- The object is grasped by the two contact points within the 2D space.
- The object is convex and rigid.
- The shape of the object is smooth at the contact points.
- The contacts on the object are the point contacts with the frictions.

As the evaluation of the object orientation, the magnitude of the contact forces is selected as the evaluation value. Grasping with less contact forces enables us to efficiently maintain the grasped posture with small grasping forces. And it avoid to break the object by means of larger forces than necessary.

2.2 Coordinate frame for analysis

The object grasped by two contact points in the 2D space is illustrated in Fig. 1(a). The object posture with respect to the gravitational direction is represented by θ in this figure. The problem is formulated in the following way: how much value of the θ should be chosen when grasping this object.



(a) task coordinate frame (b) object coordinate frame

Fig. 1. Coordinate frames.

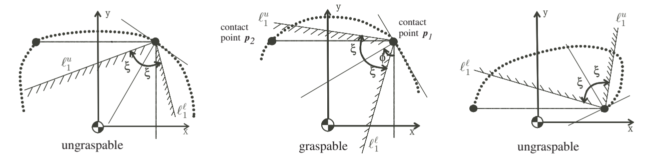


Fig. 2. Cases where the object can be grasped or not be grasped.

This problem is easily described in the object coordinate frame whose origin is set to the CoM of the object and whose y axis is defined to be orthogonal to the line connecting between two contact points. The object orientation in the task coordinate frame corresponds to the relative direction of the gravity in the object coordinate frame Fig. 1(b), i.e., the angle from the negative direction of the y axis. Note that θ is the same angle in Fig. 1(a) and (b).

In the object coordinate frame, the coordinate of the two contact points are denoted by $\boldsymbol{p}_1 = (x_1, y)^T$ and $\boldsymbol{p}_2 = (-x_2, y)^T (\neq \boldsymbol{p}_1)$, and the angle of the normal direction at the contact points from the negative direction of the y axis is ϕ_1 (CW) and ϕ_2 (CCW). Here, $x_1 > 0$, $x_2 > 0$, $y > 0$, and $0 < \phi_1 \leq \pi/2$, $0 < \phi_2 \leq \pi/2$. The contact forces are orthogonally decomposed to the the normal force N_1 or N_2 and the friction F_1 or F_2 at each contact point.

2.3 Formulation

Conditions on contact points The positional relation of the friction cones allows us to determine whether the object can be grasped by the given two contact points or not (Nguyen (1988)). If the friction cones contain the other contact point each other, then the object can be grasped by these two contact points.

This conditions is formulated as follows in the object coordinate frame. As shown in Fig. 2, the apex angle of the friction cone at the contact point $\boldsymbol{p}_1$ is set to $2\xi_1$. The object can be grasped, if and only if the angle between the upper side of the friction cone ℓ_1^u and the negative direction of the y axis is greater than $\pi/2$ as well as the angle between the lower side of the friction cone ℓ_1^ℓ and the negative direction of the y axis is less than $\pi/2$. This condition is described by the inequality

$$\phi_1 - \xi_1 < \frac{\pi}{2} < \phi_1 + \xi_1 \tag{1}$$

So is it for the contact point $\boldsymbol{p}_2$, and the condition is given as

$$\phi_2 - \xi_2 < \frac{\pi}{2} < \phi_2 + \xi_2 \tag{2}$$

These conditions can be rewritten by using the friction coefficient μ_1 and μ_2 at each contact point. The following relation holds between the apex angle and the friction coefficient.

$$\tan \xi_i = \mu_i \qquad (i = 1, 2) \tag{3}$$

Thus, subtract ϕ_1 or ϕ_2 from (1) or (2), apply tangent operation for the result, and use the relation (3). Then,

$$s_1\mu_1 + c_1 > 0, s_1\mu_1 - c_1 > 0 \tag{4}$$

is obtained from (1), and

$$s_2\mu_2 + c_2 > 0, s_2\mu_2 - c_2 > 0 \tag{5}$$

is from (2). Here, $s_1 = \sin\phi_1$, $c_1 = \cos\phi_1$, $s_2 = \sin\phi_2$, and $c_2 = \cos\phi_2$. In the following sections, the analysis is restricted to the grasp that satisfies the contact point conditions (4)-(5).

Force balance conditions In the 2D grasp, not only the force in the x and y direction but also the moment within the plane including both coordinate axes must be balanced. This balancing condition is described using matrix as

$$L\boldsymbol{F} = \boldsymbol{M} \tag{6}$$

where the contact force vector $\boldsymbol{F}$ and the gravitational vector $\boldsymbol{M}$ is defined as follows:

$$\boldsymbol{F} = [\, N_1 \ F_1 \ N_2 \ F_2 \,]^T \tag{7}$$

$$\boldsymbol{M} = [\, -Mgs \ Mgc \ 0 \,]^T \tag{8}$$

Here, M is mass of the object, g is the gravitational acceleration, $s = \sin\theta$ and $c = \cos\theta$. The matrix L is called grasp matrix, and described as

$$L = \begin{bmatrix} -s_1 & -c_1 & s_2 & c_2 \\ -c_1 & s_1 & -c_2 & s_2 \\ L_{31} & L_{32} & L_{33} & L_{34} \end{bmatrix} \tag{9}$$

$$L_{31} = -c_1x_1 + s_1y, L_{32} = s_1x_1 + c_1y$$

$$L_{33} = c_2x_2 - s_2y, L_{34} = -s_2x_2 - c_2y$$

Grasp conditions This paper has interest in the grasp with frictions. In the grasp illustrated in Fig. 3(a), the friction force is not crucial for lifting up the object, because the object is just overridden onto the two contact points. These cases can be excluded by limiting the range of the gravitational direction θ to

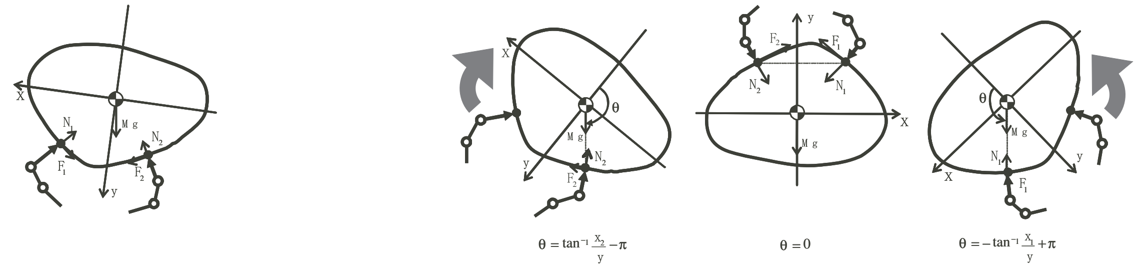
$$\theta_{min} = -\pi + \tan^{-1}\frac{x_2}{y} < \theta < \pi - \tan^{-1}\frac{x_1}{y} = \theta_{max} \tag{10}$$

as shown in Fig. 3(b). This is necessary condition for grasping the object.

Unilateral conditions The normal force works to push, not pull, the object. From the definition of the normal force direction, this condition is described as

$$N_1 > 0 \tag{11}$$

$$N_2 > 0 \tag{12}$$



(a) not grasped, jut put on

(b) object is grasped

Fig. 3. Cases where the object can be grasped or not be grasped. The range in the case (b) is considered.

Coulomb conditions To keep the contact without slipping, tangential forces cannot become greater than the maximum static friction force. It holds if $|F_i| < \mu_i N_i \, (i = 1, 2)$. This condition can be decomposed to the following four inequalities.

$$\mu_1 N_1 - F_1 > 0 \quad (13)$$

$$\mu_2 N_2 - F_2 > 0 \quad (14)$$

$$\mu_1 N_1 + F_1 > 0 \quad (15)$$

$$\mu_2 N_2 + F_2 > 0 \quad (16)$$

2.4 Problem description

The problem is defined as follows:

Definition 1. Under the contact point condition (4)-(5), find θ within the range (10) that minimize the norm of the contact force vector $\boldsymbol{F}$ such that satisfies the force balance condition (6) expressed by the equations as well as the contact force conditions (11)-(16) expressed by the inequalities.

3. ANALYSIS

3.1 Procedure

The problem defined in the above section is one class of the nonlinear optimization problems. It seemed to be easily solved by deriving the KKT conditions from the Lagrange function. However, this problem is not so, because the contact force vector $\boldsymbol{F}$ is not a direct optimizing parameter, but should be indirectly optimized by use of the object orientation θ. This is a reason why the straightforward method is not available. Thus, the following steps are taken:

(i) Parameterization of equality solution: The solution of the force balance condition given by equality (6) can be described using a parameter α. Selecting the expressions appropriately, α can express an amount that is proportional to the magnitude of the internal force.

(ii) Lower limit calculation of parameter satisfying each inequality: The parameter α must be selected so that all the inequalities conditions (11)-(16) should hold. First of all, the lower limit of the parameter α that satisfies each one of the inequality (11)-(16) is calculated among the solutions of the force balance equation. They are denoted by $\alpha_i(\theta)$ $(i = 1, \cdots, 6)$, respectively.

(iii) Piecewise analysis on lower limit of parameter satisfying all the inequalities: Next, the function $\alpha_{min}(\theta)$ is defined by finding the minimal parameter α greater than all the $\alpha_i(\theta)$ $(i = 1, \cdots, 6)$ for each posture θ. This is equivalent to selecting the maximal value among α_i $(i = 1, \cdots, 6)$ in the piecewise range of the θ. Now, the function $\alpha_{min}(\theta)$ satisfies the equalities (6) as well as all the inequalities (11)-(16).

(iv) Detection of the minimal parameters: The smaller α is, the shorter the norm of the contact force vector $\boldsymbol{F}$ is, because this norm is a monotonic increasing function of the parameter α. Thus, the minimal point of the function $\alpha_{min}(\theta)$ is calculated that corresponds to the optimal orientation of the grasped object.

3.2 Parameterization of equality solution

The general solution of the equality (6) is given as

$$\boldsymbol{F} = L^{\dagger}\boldsymbol{M} + (I - L^{\dagger}L)\boldsymbol{\kappa} \quad (17)$$

where $L^{\dagger}$ is a pseudo-inverse matrix of L that can be calculated as $L^{\dagger} = L^T(LL^T)^{-1}$, and $\boldsymbol{\kappa} \in R^4$ is an arbitrary vector. Let the first term of the right hand side to be $\boldsymbol{F}_T$. Then $\boldsymbol{F}_T$ is given from the definition as follows:

$$\boldsymbol{F}_T = \frac{Mg}{2(x_1 + x_2)} \begin{bmatrix} (x_1 + x_2)s_1 s + 2(ys - x_2 c)c_1 \\ (x_1 + x_2)c_1 s - 2(ys - x_2 c)s_1 \\ -(x_1 + x_2)s_2 s - 2(ys + x_1 c)c_2 \\ -(x_1 + x_2)c_2 s + 2(ys + x_1 c)s_2 \end{bmatrix} \quad (18)$$

On the other hand, the second term of the right hand side is a vector that exists in Ker$L$, the kernel space of the matrix L. The dimension of the Ker$L$ is one since $rank\, L = 3$. Thus, the second term is written as follows:

$$(I - L^{\dagger}L)\boldsymbol{\kappa} = \alpha \boldsymbol{F}_N \quad (19)$$

$$\boldsymbol{F}_N = \frac{Mg}{2(x_1 + x_2)} \begin{bmatrix} s_1 \\ c_1 \\ s_2 \\ c_2 \end{bmatrix} \quad (20)$$

Here, α is a scalar value that is proportional to the magnitude of the internal force exerted to the object, and $\boldsymbol{F}_N$ is a vector that satisfies the next two equations.

$$L\boldsymbol{F}_N = \boldsymbol{0} \quad (21)$$

$$\boldsymbol{F}_T^T \boldsymbol{F}_N = 0 \quad (22)$$

From the definition of this section, the solution of the equality (6) is expressed, using $\boldsymbol{F}_T$ and $\boldsymbol{F}_N$, as

$$\boldsymbol{F} = \boldsymbol{F}_T + \alpha \boldsymbol{F}_N \tag{23}$$

3.3 Lower limit of each inequality solution

Next, the function $\alpha_i(\theta)$ $(i = 1, \cdots, 6)$ defined in the section 3.1 is calculated.

(a) $N_1 > 0$

From (23), N_1 is given as

$$N_1 = \frac{Mg}{2(x_1+x_2)}\left((x_1+x_2)s_1 s - 2(x_2 c - ys)c_1 + \alpha s_1\right) \tag{24}$$

From this result, the range of α that satisfies the inequality (11) becomes

$$\alpha > \alpha_1(\theta) \tag{25}$$

$$\alpha_1(\theta) = -(x_1+x_2)s + 2(x_2 c - ys)\cot\phi_1 \tag{26}$$

(b) $N_2 > 0$

From the similar calculations, the following result is obtained.

$$\alpha > \alpha_2(\theta) \tag{27}$$

$$\alpha_2(\theta) = (x_1+x_2)s + 2(x_1 c + ys)\cot\phi_2 \tag{28}$$

(c) $\mu_1 N_1 - F_1 > 0$

From (23), F_1 is given as

$$F_1 = \frac{Mg}{2(x_1+x_2)}\left((x_1+x_2)c_1 s - 2(ys - x_2 c)s_1 + \alpha c_1\right) \tag{29}$$

From this result, the range of α that satisfies the inequality (13) becomes

$$\alpha > \alpha_3(\theta) \tag{30}$$

$$\begin{aligned}\alpha_3(\theta) &= -(x_1+x_2)s + 2(x_2 c - ys)\frac{c_1\mu_1 + s_1}{s_1\mu_1 - c_1} \\ &= -(x_1+x_2)s - 2(x_2 c - ys)\tan(\phi_1 + \xi_1)\end{aligned} \tag{31}$$

In the above calculation, the next relation is used.

$$\begin{aligned}-\frac{c_i\mu_i \pm s_i}{s_i\mu_i \mp c_i} &= \frac{s_i \pm c_i\mu_i}{c_i \mp s_i\mu_i} = \frac{\frac{s_i}{c_i} \pm \mu_i}{1 \mp \frac{s_i}{c_i}\mu_i} \\ &= \frac{\tan\phi_i \pm \tan\xi_i}{1 \mp \tan\phi_i\tan\xi_i} = \tan(\phi_i \pm \xi_i)\end{aligned} \tag{32}$$

Here, $i = 1, 2$.

(d) $\mu_2 N_2 - F_2 > 0$

From the similar calculations, the following result is obtained.

$$\alpha > \alpha_4(\theta) \tag{33}$$

$$\begin{aligned}\alpha_4(\theta) &= (x_1+x_2)s + 2(x_1 c + ys)\frac{c_2\mu_2 + s_2}{s_2\mu_2 - c_2} \\ &= (x_1+x_2)s - 2(x_1 c + ys)\tan(\phi_2 + \xi_2)\end{aligned} \tag{34}$$

(e) $\mu_1 N_1 + F_1 > 0$

From the similar calculations, the following result is obtained.

$$\alpha > \alpha_5(\theta) \tag{35}$$

$$\begin{aligned}\alpha_5(\theta) &= -(x_1+x_2)s + 2(x_2 c - ys)\frac{c_1\mu_1 - s_1}{s_1\mu_1 + c_1} \\ &= -(x_1+x_2)s - 2(x_2 c - ys)\tan(\phi_1 - \xi_1)\end{aligned} \tag{36}$$

(f) $\mu_2 N_2 + F_2 > 0$

From the similar calculations, the following result is obtained.

$$\alpha > \alpha_6(\theta) \tag{37}$$

$$\begin{aligned}\alpha_6(\theta) &= (x_1+x_2)s + 2(x_1 c + ys)\frac{c_2\mu_2 - s_2}{s_2\mu_2 + c_2} \\ &= (x_1+x_2)s - 2(x_1 c + ys)\tan(\phi_2 - \xi_2)\end{aligned} \tag{38}$$

3.4 Piecewise analysis on the lower limit of all inequalities

The function $\alpha_{min}(\theta)$ greater than all the $\alpha_i(\theta)$ $(i = 1, \cdots, 6)$ is calculated. Then, the following relation holds.

Lemma 1. Let $\theta_1 = \arctan2(x_2, y) > 0$. $\alpha_5(\theta) < \alpha_1(\theta) < \alpha_3(\theta)$ holds in the range $\theta_{min} = -\pi + \theta_1 < \theta < \theta_1$, while $\alpha_3(\theta) < \alpha_1(\theta) < \alpha_5(\theta)$ does in the range $\theta_1 < \theta < \theta_{max}$.

Note that $\arctan2(Y, X)$ returns $\tan^{-1}(Y/X)$ in the range $[-\pi, \pi]$.

Proof 1. Calculating $\alpha_3 - \alpha_1$ (or $\alpha_5 - \alpha_1$), the following equation is obtained

$$\begin{aligned}&2(x_2 c - ys)\left(\frac{c_1\mu_1 \pm s_1}{s_1\mu_1 \mp c_1} - \frac{c_1}{s_1}\right) \\ &= \mp\frac{2}{s_1(s_1\mu_1 \mp c_1)}\sqrt{x_2^2 + y^2}\sin(\theta - \theta_1)\end{aligned} \tag{39}$$

Here, $0 < \phi_1 < \pi/2$, the numerator is positive because of (4), $\sin(\theta - \theta_1) > 0$ in $\theta_1 < \theta < \theta_{max}$ and $\sin(\theta - \theta_1) < 0$ in $\theta_{min} < \theta < \theta_1$. Thus, the lemma is proved.

The next lemma also holds from the similar calculations.

Lemma 2. Let $\theta_2 = \arctan2(x_1, y) > 0$. $\alpha_4(\theta) < \alpha_2(\theta) < \alpha_6(\theta)$ holds in the range $\theta_{min} < \theta < -\theta_2$, while $\alpha_6(\theta) < \alpha_2(\theta) < \alpha_4(\theta)$ holds in the range $-\theta_2 < \theta < \pi - \theta_2 = \theta_{max}$.

From the above two lemmas, the magnitude of next two α_i have only to be compared within the following three range: α_3 versus α_6 in the range $\theta_{min} < \theta < -\theta_2$, α_4 versus α_5 in the range $\theta_1 < \theta < \theta_{max}$, and α_3 versus α_4 in the range $-\theta_2 < \theta < \theta_1$. Regarding to these relations, the following lemmas are satisfied.

Lemma 3. There exists a θ_{36} such that $\alpha_6(\theta) < \alpha_3(\theta)$ in the range $\theta_{36} < \theta < \theta_{36} + \pi$, while $\alpha_3(\theta) < \alpha_6(\theta)$ in the range $\theta_{36} - \pi < \theta < \theta_{36}$. This θ_{36} is given as follows:

$$\theta_{36} = \arctan2(B_{36}, A_{36}) \tag{40}$$

where

$$A_{36} = y\left(\tan(\phi_1 + \xi_1) + \tan(\phi_2 - \xi_2)\right) - (x_1 + x_2) \tag{41}$$

$$B_{36} = x_2\tan(\phi_1 + \xi_1) - x_1\tan(\phi_2 - \xi_2) \tag{42}$$

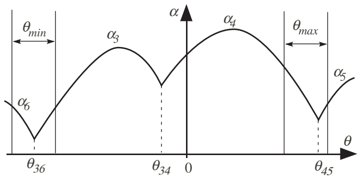


Fig. 4. a rough sketch graph of minimal α vs. θ.

Lemma 4. There exists a θ_{45} such that $\alpha_4(\theta) < \alpha_5(\theta)$ in the range $\theta_{45} < \theta < \theta_{45} + \pi$, while $\alpha_5(\theta) < \alpha_4(\theta)$ in the range $\theta_{45} - \pi < \theta < \theta_{45}$. This θ_{45} is given as follows:

$$\theta_{45} = \text{arctan2}(B_{45}, A_{45}) \tag{43}$$

where

$$A_{45} = y\,(\tan(\phi_1 - \xi_1) + \tan(\phi_2 + \xi_2)) - (x_1 + x_2) \tag{44}$$

$$B_{45} = x_2 \tan(\phi_1 - \xi_1) - x_1 \tan(\phi_2 + \xi_2) \tag{45}$$

Lemma 5. There exists a θ_{34} such that $\alpha_3(\theta) < \alpha_4(\theta)$ in the range $\theta_{34} < \theta < \theta_{34} + \pi$, while $\alpha_4(\theta) < \alpha_3(\theta)$ in the range $\theta_{34} - \pi < \theta < \theta_{34}$. This θ_{34} is given as follows:

$$\theta_{34} = \text{arctan2}(B_{34}, A_{34}) \tag{46}$$

where

$$A_{34} = (x_1 + x_2) - y\,(\tan(\phi_1 + \xi_1) + \tan(\phi_2 + \xi_2)) \tag{47}$$

$$B_{34} = x_1 \tan(\phi_2 + \xi_2) - x_2 \tan(\phi_1 + \xi_1) \tag{48}$$

The proof for the lemma 5 is shown below:

Proof 2. From (31) and (34),

$$\begin{aligned}
&\alpha_4 - \alpha_3 \\
&= 2\{(x_1 + x_2) - y\,(\tan(\phi_1 + \xi_1) + \tan(\phi_2 + \xi_2))\} \sin\theta \\
&\quad -2\{x_1 \tan(\phi_2 + \xi_2) - x_2 \tan(\phi_1 + \xi_1)\} \cos\theta \\
&= 2(A_{34} \sin\theta - B_{34} \cos\theta) \\
&= 2\sqrt{A_{34}^2 + B_{34}^2}\, \sin(\theta - \theta_{34})
\end{aligned} \tag{49}$$

It is trivial from the above equation.

3.5 Detection of the minimal parameters

The parameter α_{min} that satisfies equality (6) as well as all the inequalities (11)-(16) becomes a function of the θ, and now it is given as the following equations.

$$\alpha_{min}(\theta) = \begin{cases} \alpha_6(\theta) \ (\theta \le \theta_{36}) \\ \alpha_3(\theta) \ (\theta_{36} \le \theta \le \theta_{34}) \\ \alpha_4(\theta) \ (\theta_{34} \le \theta \le \theta_{45}) \\ \alpha_5(\theta) \ (\theta_{45} \le \theta) \end{cases} \tag{50}$$

Accordingly, the orientations that could be the minimal point are: θ_{36}, θ_{34}, and θ_{45}, as shown in Fig. 4. The relation $\theta_{36} < \theta_{34} < \theta_{45}$ can be obtained graphically from the configuration of two friction cones.

Then, what is the physical situation in these minimal point? From the result of the analysis on the point θ_{34}, the next lemma is obtained.

Lemma 6. When the gravitational direction is given as θ_{34} in the object coordinate frame, the intersection point of two upper sides of friction cone is faced in the opposite direction of the gravity with respect to the CoM of the object in the task coordinate frame.

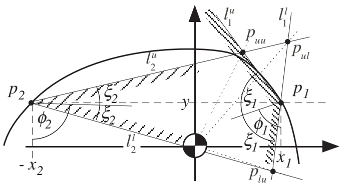


Fig. 5. relative orientation of friction-cones' intersection point.

Proof 3. In Fig. 5, the upper side of the friction cone of the contact point $\boldsymbol{p}_1$ is represented as the line ℓ_u^1 that goes through the point (x_1, y) and whose slope is $\tan(\frac{\pi}{2} - \phi_1 - \xi_1) = \cot(\phi_1 + \xi_1)$. It is expressed by

$$Y - y = \cot(\phi_1 + \xi_1)(X - x_1) \tag{51}$$

On the other hand, the upper side of the friction cone of the contact point $\boldsymbol{p}_2$ is represented as the line ℓ_u^2 that goes through the point $(-x_2, y)$ and whose slope is $\tan(\phi_2 + \xi_2 - \frac{\pi}{2}) = -\cot(\phi_2 + \xi_2)$. It is also expressed by

$$Y - y = -\cot(\phi_2 + \xi_2)(X + x_2) \tag{52}$$

The intersection point of these two lines is:

$$X = \frac{x_1 \tan(\phi_2 + \xi_2) - x_2 \tan(\phi_1 + \xi_1)}{\tan(\phi_1 + \xi_1) + \tan(\phi_2 + \xi_2)} \tag{53}$$

$$Y = \frac{y(\tan(\phi_1 + \xi_1) + \tan(\phi_2 + \xi_2)) - (x_1 + x_2)}{\tan(\phi_1 + \xi_1) + \tan(\phi_2 + \xi_2)} \tag{54}$$

Now, let $T = \tan(\phi_1 + \xi_1) + \tan(\phi_2 + \xi_2)$, then $T < 0$ from (1)-(2). Thus, the angle from the negative direction of y axis θ_c is given by

$$\begin{aligned}
\theta_c &= \text{arctan2}(X, -Y) = \text{arctan2}\left(\frac{B_{34}}{T}, -(\frac{-A_{34}}{T})\right) \\
&= \text{arctan2}(-B_{34}, -A_{34}) = \theta_{34} \pm \pi
\end{aligned} \tag{55}$$

Namely, the orientation of the friction cone intersection point is just the opposite direction of the gravitational direction θ_{34}.

In the similar way, the following lemmas holds.

Lemma 7. Let $\boldsymbol{p}_{u\ell}$ the intersection point of ℓ_1^u (the upper side of friction cone at the contact point $\boldsymbol{p}_1$) and ℓ_2^ℓ (the lower side of friction cone at the contact point $\boldsymbol{p}_2$). If the gravitational direction is given as θ_{36} in the object coordinate frame, then $\boldsymbol{p}_{u\ell}$ and the CoM of the object aligns in the gravitational direction in the task coordinate frame.

Lemma 8. Let $\boldsymbol{p}_{\ell u}$ the intersection point of ℓ_1^ℓ (the lower side of friction cone at the contact point $\boldsymbol{p}_1$) and ℓ_2^u (the upper side of friction cone at the contact point $\boldsymbol{p}_2$). If the gravitational direction is given as θ_{45} in the object coordinate frame, then $\boldsymbol{p}_{\ell u}$ and the CoM of the object aligns in the gravitational direction in the task coordinate frame.

These results are illustrated in Fig. 6. These could be three possible minimal points: θ_{36}, θ_{34}, and θ_{45} though whether

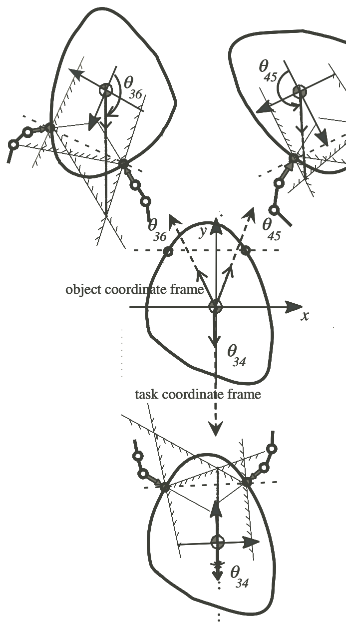


Fig. 6. Object orientations that could give the local minimal point.

θ_{36} and θ_{45} are in the range (10) or not must be discussed more in our future works. However, θ_{36}, θ_{45}, θ_{max} and θ_{min} is a solution where the object is grasped from the side. Accordingly, the meaningful solution is only θ_{34} where the object is grasped from above.

The analysis in this section is summarized to the next theorem:

Theorem 1. Consider a 2D grasp where a convex rigid object is grasped by two contact points with frictions. The norm of the contact force is a locally minimal at the posture where the intersection point of the upper sides of each friction cones at two contact points is located in the opposite direction of the gravity from the CoM of the object.

Unfortunately, the grasp with the locally minimal contact force norm does not possess enough robustness: slight posture deviation will make a slip on the grasped object. However, this result indicates that: when the gripping forces are limited, we should try several object orientations. Then, we may be able to lift it up by grasping it from the upper side. (Of course, the gripping forces must be sufficient then.)

4. CONCLUSIONS

In this paper, a 2D grasp is considered where a convex rigid object is grasped by two contact points with frictions. Such a grasp required some equality conditions, i.e., the force balance conditions, as well as some inequality conditions, i.e., the contact point conditions and the contact force conditions consisting of the unilateral and Coulomb conditions. The purpose of this paper is to find the posture that minimize the norm of the contact force vector whose element is composed from the normal and friction force at each contact point.

In the analysis, the solution of the equality conditions are parameterized at first. Then, the minimal parameter values are calculated by the piecewise analysis since the norm of the contact force vector becomes monotonic increasing function of this parameter.

Using the relation between the friction coefficient and the apex angle of the friction cone effectively, the following fact becomes clear: the norm of the contact force takes a local minimal value at the posture where the intersection point of the upper sides of each friction cones at two contact points is located in the opposite direction of the gravity from the CoM of the object. This fact coincides our numerical analysis on the grasp of the circular object (Ito et al. (2007)). As a future works, we extend this analysis to the 3D grasp, and to the effective manipulation of the object requiring less contact forces.

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